

Upper bounds on the charge susceptibility of many-electron systems coupled to the quantized radiation field

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Abstract

We extend the Kubo-Kishi theorem concerning the charge susceptibility of the Hubbard model in the following way: (i) The electron-photon interaction is taken into account. (ii) Not only on-site but also general Coulomb repulsions are considered.

1 Introduction and the results

In order to study the origin of ferromagnetism, Gutzwiller [8], Hubbard [10] and Kanamori [11] proposed a tight binding model of electrons with on-site Coulomb interaction, which is called the Hubbard model nowadays. If there is no Coulomb interaction, the model demonstrates paramagnetism due to the Pauli exclusion principle. Therefore, in 1960's, researchers had a common question whether it is possible to transform the paramagnetism to ferromagnetism by taking the Coulomb interaction into account. The first rigorous result about ferromagnetism in the Hubbard model was given by Nagaoka [17]. He showed that ferromagnetism appears in the ground state when there is exactly a single hole and the strength of the Coulomb interaction is infinite. Since then, the Hubbard model has been widely recognized as a simplified model for rigorous study of ferromagnetism. For a review of the rigorous results, we refer to [20].

In 1989, Lieb showed that the Hubbard model on a connected bipartite lattice at half filling exhibits ferrimagnetism by applying the spin reflection positivity [13]. In 1990, Kubo and Kishi [12] showed the absence of charge long range order in the Hubbard system on the bipartite lattice by using the method of Dyson, Lieb and Simon [4]. Their result can be regarded as a finite temperature version of Lieb's theorem.

In this paper, we rigorously study a crystal coupled to the quantized radiation field. Our Hamiltonian is the Hubbard model with an electron-photon interaction term. Besides, not only on-site but also general Coulomb repulsions are considered. The purpose of this paper is to extend the Kubo-Kishi theorem to such a generalized case. We note that the influence of the classical electromagnetic field was studied by Lieb [14]. In contrast we take account of effects of the quantized electromagnetic field. Also we remark that a similar model was considered in [5, 6]. The effects of electron-photon interaction in graphen were investigated by an exact renormalization group analysis. In this paper, we will not restrict ourself to the honeycomb lattice. Our method can be extend to general bipartite lattices straightforwardly.

1.1 Background: The Hubbard model

First of all, we review some known results for the Hubbard model. For each $d \in \{1, 2, 3\}$ and $\ell \in \mathbb{N}$ even, let

$$\Lambda = [-\ell/2, \ell/2]^d \cap \mathbb{Z}^d. \quad (1.1)$$

In what follows, we regard Λ as a simple cubic torus. The Hubbard model is given by

$$H_H = \sum_{\substack{x, y \in \Lambda, |x-y|=1 \\ \sigma=\uparrow, \downarrow}} (-t) c_{x\sigma}^* c_{y\sigma} + \frac{1}{2} \sum_{x \in \Lambda} U_0 (n_x - \mathbb{1})(n_x - \mathbb{1}). \quad (1.2)$$

H_H lives in the Hilbert space

$$\mathfrak{E} = \mathfrak{F}_a(\ell^2(\Lambda) \oplus \ell^2(\Lambda)), \quad (1.3)$$

where $\mathfrak{F}_a(\mathfrak{h})$ is the fermionic Fock space over \mathfrak{h} defined by $\mathfrak{F}_a(\mathfrak{h}) = \bigoplus_{n=0}^{\infty} \wedge^n \mathfrak{h}$. $\wedge^n \mathfrak{h}$ is the n -fold anti-symmetric tensor product of \mathfrak{h} . $c_{x\sigma}$ and $c_{x\sigma}^*$ are the fermionic annihilation- and creation operators respectively. The following anti-commutation relations hold:

$$\{c_{x\sigma}, c_{x'\sigma'}^*\} = \delta_{\sigma\sigma'} \delta_{xx'}, \quad (1.4)$$

$$\{c_{x\sigma}, c_{x'\sigma'}\} = 0 = \{c_{x\sigma}^*, c_{x'\sigma'}^*\}. \quad (1.5)$$

$n_x = n_{x\uparrow} + n_{x\downarrow}$ is the number operator at vertex x with $n_{x\sigma} = c_{x\sigma}^* c_{x\sigma}$. U_0 is the strength of the on-site Coulomb interaction. $t > 0$ is the nearest-neighbor hopping integral.

Let $\delta n_x := n_x - \mathbb{1}$. Set

$$\widetilde{\delta n}_p = |\Lambda|^{-1/2} \sum_{x \in \Lambda} e^{-ix \cdot p} \delta n_x. \quad (1.6)$$

The charge susceptibility is defined by

$$\chi_{\beta, H}(p) = \lim_{\ell \rightarrow \infty} \beta (\widetilde{\delta n}_{-p}, \widetilde{\delta n}_p)_{\beta, \Lambda, H}, \quad (1.7)$$

where $(\cdot, \cdot)_{\beta, \Lambda, H}$ is the Duhamel two-point function given by

$$(A, B)_{\beta, \Lambda, H} = Z_{\beta, \Lambda, H}^{-1} \int_0^1 ds \operatorname{Tr} \left[e^{-s\beta H_H} A e^{-(1-s)\beta H_H} B \right], \quad (1.8)$$

$$Z_{\beta, \Lambda, H} = \operatorname{Tr} [e^{-\beta H_H}]. \quad (1.9)$$

We employ the thermal average with respect to the grand canonical Gibbs state at inverse temperature β . For any β and Λ , we can check that the thermal average density of the system satisfies $\langle n_o \rangle_{\beta, \Lambda, H} := Z_{\beta, \Lambda, H}^{-1} \operatorname{Tr} [n_o e^{-\beta H_H}] = 1$, that is, the system at half-filling is considered.

In [12], Kubo and Kishi proved the following:

Theorem 1.1 *Assume $U_0 > 0$. Then one has*

$$\chi_{\beta, H}(p) \leq U_0^{-1} \quad (1.10)$$

for all $p \in [-\pi, \pi]^d$.

Using the Falk-Bruch inequality and Theorem 1.1, we conclude the absence of the charge long-range order.

1.2 Many-electron system coupled to the radiation field

For $L \in \mathbb{N}$ even, set

$$V = [-L/2, L/2]^3, \quad V^* = \left(\frac{2\pi}{L}\mathbb{Z}\right)^3. \quad (1.11)$$

In this paper, we always assume that $L \geq \ell$. Thus we can regard Λ as a subset of V by the following manner:

$$\Lambda \equiv \begin{cases} \{(x, 0, 0) \in V \mid x \in \Lambda\} & d = 1 \\ \{(x, 0) \in V \mid x \in \Lambda\} & d = 2 \\ \Lambda & d = 3. \end{cases} \quad (1.12)$$

We are interested in the interacting electron-photon system. The total Hamiltonian is

$$H = \sum_{\substack{x, y \in \Lambda, \\ \sigma = \uparrow, \downarrow}} (-t) \exp \left\{ ie \int_{C_{xy}} dr \cdot A(r) \right\} c_{x\sigma}^* c_{y\sigma} \quad (1.13)$$

$$+ \frac{1}{2} \sum_{x, y \in \Lambda} U(x - y)(n_x - \mathbb{1})(n_y - \mathbb{1}) + \sum_{\lambda=1,2} \sum_{k \in V^*} \omega(k) a(k, \lambda)^* a(k, \lambda) \quad (1.14)$$

$$=: H_{e-p} + H_{e-e} + H_f. \quad (1.15)$$

H acts in the Hilbert space

$$\mathfrak{E} \otimes \mathfrak{F}, \quad (1.16)$$

where $\mathfrak{F} = \mathfrak{F}_s(L^2(V^* \times \{1, 2\}))$. $\mathfrak{F}_s(\mathfrak{h})$ is the bosonic Fock space defined by $\mathfrak{F}_s(\mathfrak{h}) = \oplus_{n=0}^{\infty} \otimes_s^n \mathfrak{h}$, where $\otimes_s^n \mathfrak{h}$ is the n -fold symmetric tensor product of \mathfrak{h} . $a(k, \lambda)$ and $a(k, \lambda)^*$ are the bosonic annihilation- and creation operators respectively. These satisfy the canonical commutation relations:

$$[a(k, \lambda), a(k', \lambda')^*] = \delta_{\lambda\lambda'} \delta_{kk'}, \quad (1.17)$$

$$[a(k, \lambda), a(k', \lambda')] = 0 = [a(k, \lambda)^*, a(k', \lambda')^*]. \quad (1.18)$$

The quantized vector potential is given by

$$A(x) = |V|^{-1/2} \sum_{\lambda=1,2} \sum_{k \in V^*} \frac{\chi_\kappa(k)}{\sqrt{2\omega(k)}} \varepsilon(k, \lambda) \left(a(k, \lambda) e^{ik \cdot x} + a(k, \lambda)^* e^{-ik \cdot x} \right). \quad (1.19)$$

The form factor χ_κ is the indicator function of the ball of radius $\kappa < \infty$. The dispersion relation is chosen to be $\omega(k) = |k|$ for $k \in V^* \setminus \{0\}$, $\omega(0) = m_0$ with $0 < m_0 < 2\pi/L$.¹ C_{xy} is a piecewise smooth curve from x to y . The polarization vectors are denoted by $\varepsilon(k, \lambda) = (\varepsilon_1(k, \lambda), \varepsilon_2(k, \lambda), \varepsilon_3(k, \lambda))$, $\lambda = 1, 2$. Together with $k/|k|$, they form a basis, which for concreteness is taken as

$$\varepsilon(k, 1) = \frac{(k_2, -k_1, 0)}{\sqrt{k_1^2 + k_2^2}}, \quad \varepsilon(k, 2) = \frac{k}{|k|} \wedge \varepsilon(k, 1). \quad (1.20)$$

¹The modification at $k = 0$ is needed in order to guarantee that $e^{-\beta H}$ is trace class. However when we take the limit $\ell \rightarrow \infty$ at the final stage of our arguments, our conclusions are independent of the value of m_0 .

For convenience, we put $\varepsilon(k, \lambda) = 0$ if $(k_1, k_2) = (0, 0)$. The charge of a single electron is denoted by e . $U(x)$ is a real-valued function on \mathbb{Z}^d such that $\|U\|_\infty := \max_{x \in \mathbb{Z}^d} |U(x)| < \infty$ and $U(-x) = U(x)$. It is trivial that H is self-adjoint and bounded from below.

We assume the following:

(A. 1) $U \in \ell^1(\mathbb{Z}^d)$.

(A. 2) For all $\ell > 0$, it holds that $\hat{U}_\Lambda(p) \geq 0$, where $\hat{f}_\Lambda(p) = \sum_{x \in \Lambda} e^{-ix \cdot p} f(x)$.

As before the charge susceptibility is defined by

$$\chi_\beta(p) = \lim_{\ell \rightarrow \infty} \beta (\widetilde{\delta n_{-p}}, \widetilde{\delta n_p})_{\beta, \Lambda}, \quad (1.21)$$

where $(\cdot, \cdot)_{\beta, \Lambda}$ is the Duhamel two-point function associated with H .

The following theorem is an extension of Theorem 1.1.

Theorem 1.2 *Assume (A. 1) and (A. 2). For each $p \in [-\pi, \pi]^d$ such that $\hat{U}(p) > 0$, one has*

$$\chi_\beta(p) \leq \hat{U}(p)^{-1}. \quad (1.22)$$

Corollary 1.3 *Assume (A. 1) and (A. 2). In addition, assume that there exists a constant $u_0 > 0$ such that $\hat{U}(p) \geq u_0$ for all $p \in [-\pi, \pi]^d$. Then one has*

$$\chi_\beta(p) \leq u_0^{-1}. \quad (1.23)$$

Thus there is no charge long-range order.

Example 1 For each $U_0 > 0$, let $U(x) = U_0 \delta_{xo}$, where δ_{xy} is the Kronecker delta. Then **(A. 1)** and **(A. 2)** are fulfilled with $\hat{U}_\Lambda(p) = U_0$. For all $U_0 > 0$, there is no charge long-range order. \diamond

Example 2 For each $U_0, U_1 \geq 0$, we define

$$U(x) = \begin{cases} U_0 & x = o \\ U_1/2d & |x| = 1 \\ 0 & \text{otherwise} \end{cases}. \quad (1.24)$$

Then we see that $\hat{U}_\Lambda(p) = (U_0 - U_1) + \frac{U_1}{d} \sum_{j=1}^d (1 + \cos p_j)$. Clearly **(A. 1)** and **(A. 2)** are satisfied provided $U_0 \geq U_1$. There is no charge long-range order provided that $U_0 > U_1$ by Corollary 1.3. On the other hand, if $U_0 = U_1$, then $\chi_\beta(p)$ could diverge at extreme points of $[-\pi, \pi]^d$. \diamond

1.3 Organization of the paper

The rest of this paper is organized as follows. In Section 2, as a preliminary section, we construct a trace formula for the quantized electromagnetic field. Then, in Section 3, we will prove Theorem 1.2.

2 Preliminaries

In [15], the Schrödinger representation of the radiation field is constructed. On the other hand, Arai presented some trace formulas associated with Gibbs states by introducing Euclidean bose field [1, 2]. By combining the ideas in [1, 2, 15], we will derive a trace formula concerning the photon field in this section.

Let $q(\cdot, \cdot)$ be a bilinear form from $\oplus^3 L^2(V) \times \oplus^3 L^2(V)$ to \mathbb{C} defined by

$$q(f, g) = \frac{1}{2} |V|^{-1} \sum_{k \in V^*} \langle \hat{f}(k), (\mathbb{1}_3 - |\tilde{k}\rangle\langle\tilde{k}|) \hat{g}(k) \rangle_{\mathbb{C}^3}, \quad f, g \in \oplus^3 L^2(V), \quad (2.1)$$

where $\tilde{k} = k/|k|$ if $(k_1, k_2) \neq 0$, $\tilde{k} = 0$ otherwise. $\hat{f}(k)$ means $\hat{f}(k) = (\hat{f}_1(k), \hat{f}_2(k), \hat{f}_3(k))$ with

$$\hat{f}_j(k) = \frac{1}{\sqrt{|V|}} \int_V dx f_j(x) e^{-ix \cdot k}. \quad (2.2)$$

Since $\mathbb{1}_3 - |\tilde{k}\rangle\langle\tilde{k}|$ is a positive semidefinite matrix in \mathbb{C}^3 , one sees that $q(f, f) \geq 0$. However q is degenerate. Namely $q(f, f) = 0$ does not imply $f = 0$. Let $\mathcal{N} = \{f \in \oplus^3 L^2(V) \mid q(f, f) = 0\}$. Now we define a Hilbert space by

$$\mathcal{H}(V) = \overline{\oplus^3 L^2(V) / \mathcal{N}}^q. \quad (2.3)$$

By definition, the inner product of $\mathcal{H}(V)$ satisfies

$$\langle [f], [g] \rangle_{\mathcal{H}(V)} = q(f, g), \quad f, g \in \oplus^3 L^2(V), \quad (2.4)$$

where $[f] \in \oplus^3 L^2(V) / \mathcal{N}$ is the equivalence class of f . Henceforth we denote $[f]$ by f if no confusion occurs.

Let $U_t, t \geq 0$ be an operator defined by

$$\widehat{U_t f}(k) = e^{-t\omega(k)} \hat{f}(k), \quad f \in \oplus^3 L^2(V), \quad k \in V^*. \quad (2.5)$$

Since $\|U_t f\|_{\mathcal{H}(V)} \leq \|f\|_{\mathcal{H}(V)}$ for $f \in \oplus^3 L^2(V)$, one can extend U_t to a bounded operator on $\mathcal{H}(V)$. We also denote it by the same symbol. It is not so hard to see that U_t is a strongly continuous one-parameter semigroup. Thus there is a self-adjoint operator $\tilde{\omega}$ such that $U_t = e^{-t\tilde{\omega}}$. For each $f \in \oplus^3 L^2(V) \cap \text{dom}(\tilde{\omega})$, we have that $\widehat{\tilde{\omega} f}(k) = \omega(k) \hat{f}(k)$. If $\gamma > 3$, then $\sum_{k \in V^*} \omega(k)^{-\gamma}$ is finite. Hence $\text{Tr}_{\mathcal{H}(V)}[\tilde{\omega}^{-\gamma}]$ is finite as well.

Let $\{\phi(f) \mid f \in \mathcal{H}(V)\}$ be the Gaussian random process indexed by $\mathcal{H}(V)$. We denote by $(\mathcal{Q}, \mathcal{B}, \nu)$ the underlying probability space of the process.

For each $s \in \mathbb{R}$, we define an inner product $(\cdot, \cdot)_s$ on $\text{dom}(\tilde{\omega}^s)$ by

$$(f, g)_s := \left\langle \tilde{\omega}^{s/2} f, \tilde{\omega}^{s/2} g \right\rangle_{\mathcal{H}(V)}, \quad f, g \in \text{dom}(\tilde{\omega}^s). \quad (2.6)$$

For $s \geq 0$, $\mathcal{H}_s(V) = (\text{dom}(\tilde{\omega}^s), (\cdot, \cdot)_s)$ becomes a Hilbert space. For $s < 0$, we denote by $\mathcal{H}_s(V)$ the completion of $\mathcal{H}(V)$ in the norm $\|\cdot\|_s := (\cdot, \cdot)_s^{1/2}$. For all $s \in \mathbb{R}$ the dual space of $\mathcal{H}_s(V)$ can be identified with $\mathcal{H}_{-s}(V)$ through the bilinear form $_{-s}\langle \cdot, \cdot \rangle_s$ on $\mathcal{H}_{-s}(V) \times \mathcal{H}_s(V)$ such that $_{-s}\langle f, g \rangle_s = \langle f, g \rangle_{\mathcal{H}(V)}$ for $f \in \mathcal{H}(V) \cap \mathcal{H}_{-s}(V), g \in \mathcal{H}(V) \cap \mathcal{H}_s(V)$. Fix $\gamma > 3$ arbitrarily. Since $\tilde{\omega}^{-\gamma}$ is in the trace class, we can take

$\mathcal{Q} = \mathcal{H}_{-\gamma}(V)$ and $\phi(f) = -\gamma\langle\phi, f\rangle_\gamma$ by a theorem of Gross [7]. Let $\mathcal{A}(f)$ be the multiplication operator by the function $-\gamma\langle\phi, f\rangle_\gamma$. As usual we can define the Wick product as follows:

$$:\mathcal{A}(f): = \mathcal{A}(f), \quad (2.7)$$

$$\begin{aligned} :\mathcal{A}(f_1)\cdots\mathcal{A}(f_n): &= \mathcal{A}(f_1):\mathcal{A}(f_2)\cdots\mathcal{A}(f_n): \\ &\quad - \sum_{j=2}^n q(f_1, f_j) :\mathcal{A}(f_2)\cdots\widehat{\mathcal{A}(f_j)}\cdots\mathcal{A}(f_n): \end{aligned} \quad (2.8)$$

for $f, f_1, \dots, f_n \in \oplus^3 L^2(V)$, where $\widehat{\mathcal{A}(f_j)}$ indicates the omission of $\mathcal{A}(f_j)$. For each bounded operator S on $\mathcal{H}_\gamma(V)$, its second quantization $\Gamma(S)$ is defined by

$$\Gamma(S) : \mathcal{A}(f_1)\cdots\mathcal{A}(f_n) := \mathcal{A}(Sf_1)\cdots\mathcal{A}(Sf_n) :. \quad (2.9)$$

Let T be a self-adjoint operator on $\mathcal{H}_\gamma(V)$. Then $\Gamma(e^{itT})$ becomes a strongly continuous one-parameter unitary group on $L^2(\mathcal{Q})$. Thus there exists a unique self-adjoint operator $d\Gamma(T)$ such that $\Gamma(e^{itT}) = e^{itd\Gamma(T)}$.

Set $\mathcal{A}_j(f) := \mathcal{A}(\oplus_{i=1}^3 \delta_{ij} f)$. Let ι be a unitary operator from $L^2(\mathcal{Q})$ onto \mathfrak{P} defined by

$$\iota : \mathcal{A}_{j_1}(f_1)\cdots\mathcal{A}_{j_n}(f_n) := A_{j_1}(f_1)_+ \cdots A_{j_n}(f_n)_+ \Omega_s, \quad f_1, \dots, f_n \in L^2(\mathbb{R}^3), \quad (2.10)$$

where Ω_s is the bosonic Fock vacuum and

$$A_j(f) = A_j(f)_+ + A_j(f)_-, \quad (2.11)$$

$$\left. \begin{matrix} A_j(f)_+ \\ A_j(f)_- \end{matrix} \right\} = (2|V|)^{-1/2} \sum_{\lambda=1,2} \sum_{k \in V^*} \varepsilon(k, \lambda) \hat{f}(\pm k) \begin{Bmatrix} a(k, \lambda)^* \\ a(k, \lambda) \end{Bmatrix}. \quad (2.12)$$

Then we obtain

$$\overline{\iota A_j(f)} \iota^{-1} = \mathcal{A}_j(f), \quad \iota H_f \iota^{-1} = d\Gamma(\tilde{\omega}). \quad (2.13)$$

Let $Z_\beta = \text{Tr}_{L^2(\mathcal{Q})}[e^{-\beta d\Gamma(\tilde{\omega})}]$. As is well-known, the Planck's formula holds:

$$Z_\beta = \prod_{k \in V^*} \frac{1}{1 - e^{-\beta \omega(k)}}. \quad (2.14)$$

For $\beta > 0$, let $\mathcal{Q}_\beta = C([0, \beta]; \mathcal{Q})$ be the space of \mathcal{Q} -valued continuous functions on $[0, \beta]$. For each $\Phi \in \mathcal{Q}_\beta$, we denote the value at $s \in [0, \beta]$ by Φ_s . Then there exists a probability space $(\mathcal{Q}_\beta, \mathcal{F}_\beta, \nu_\beta)$ such that $\{\Phi_s(f) \mid f \in \mathcal{H}_\gamma(V), s \in [0, \beta]\}$ is a family of Gaussian random variables on $(\mathcal{Q}_\beta, \mathcal{F}_\beta, \nu_\beta)$ with covariance

$$\int_{\mathcal{Q}_\beta} \Phi_t(f) \Phi_s(g) d\nu_\beta(\Phi) = \left\langle f, (\mathbb{1} - e^{-\beta \tilde{\omega}})^{-1} (e^{-(\beta - |t-s|)\tilde{\omega}} + e^{-|t-s|\tilde{\omega}}) g \right\rangle_{\mathcal{H}(V)}. \quad (2.15)$$

Moreover we have the following useful formula.

Lemma 2.1 *Let $F_1, \dots, F_n \in L^\infty(\mathcal{Q})$ and let $0 \leq t_1 < t_2 < \dots < t_n \leq \beta$. Then we have*

$$\begin{aligned} & \text{Tr}_{L^2(\mathcal{Q})} \left[e^{-t_1 d\Gamma(\tilde{\omega})} F_1 e^{-(t_2-t_1)d\Gamma(\tilde{\omega})} \dots e^{-(t_n-t_{n-1})d\Gamma(\tilde{\omega})} F_n e^{-(\beta-t_n)d\Gamma(\tilde{\omega})} \right] / Z_\beta \\ &= \int_{\mathcal{Q}_\beta} F_1(\Phi_{t_1}) \dots F_n(\Phi_{t_n}) d\nu_\beta(\Phi). \end{aligned} \quad (2.16)$$

Proof. See [1, 2](cf. [9]). \square

3 Proof of Theorem 1.2

Our strategy of the proof is similar to [16]. For reader's convenience, we provide a complete proof.

3.1 Rewriting the Hamiltonian

3.1.1 Expression of the Hamiltonian in $(\mathfrak{F}_a \otimes \mathfrak{F}_a) \otimes \mathfrak{P}$

Note that $\mathfrak{E} = \mathfrak{F}_a \otimes \mathfrak{F}_a$ with $\mathfrak{F}_a = \mathfrak{F}_a(\ell^2(\Lambda))$. Under this identification, we see that

$$c_{x\uparrow} = c_x \otimes \mathbb{1}, \quad c_{x\downarrow} = (-\mathbb{1})^{N_a} \otimes c_x, \quad (3.1)$$

where c_x and c_x^* are the fermionic annihilation- and creation operators on \mathfrak{F}_a , N_a is the fermionic number operator given by $N_a = \sum_{x \in \Lambda} n_x$ with $n_x = c_x^* c_x$. Thus our Hamiltonian becomes

$$H = H_{e-p} + H_{e-e} \otimes \mathbb{1} + \mathbb{1} \otimes \mathbb{1} \otimes H_f \quad (3.2)$$

where

$$H_{e-p} = -T_{+e,\uparrow} - T_{+e,\downarrow}, \quad (3.3)$$

$$T_{\pm e,\uparrow} = \sum_{x,y \in \Lambda, |x-y|=1} t c_x^* c_y \otimes \mathbb{1} \otimes \exp \left\{ \pm i e \int_{C_{xy}} dr \cdot A(r) \right\}, \quad (3.4)$$

$$T_{\pm e,\downarrow} = \sum_{x,y \in \Lambda, |x-y|=1} t \mathbb{1} \otimes c_x^* c_y \otimes \exp \left\{ \pm i e \int_{C_{xy}} dr \cdot A(r) \right\} \quad (3.5)$$

and

$$H_{e-e} = \frac{1}{2} \sum_{x,y \in \Lambda} U(x-y) (n_x \otimes \mathbb{1} + \mathbb{1} \otimes n_x - \mathbb{1}) (n_y \otimes \mathbb{1} + \mathbb{1} \otimes n_y - \mathbb{1}). \quad (3.6)$$

3.1.2 The hole-particle transformation

Note that Λ can be divided into two disjoint sets Λ_e and Λ_o , where $\Lambda_e = \{x \in \Lambda \mid x_1 + x_2 + \dots + x_d \text{ is even}\}$ and $\Lambda_o = \{x \in \Lambda \mid x_1 + x_2 + \dots + x_d \text{ is odd}\}$. The hole-particle transformation is a unitary operator \mathcal{U} on \mathfrak{E} such that

$$\mathcal{U} c_x \otimes \mathbb{1} \mathcal{U}^* = \gamma(x) c_x^* \otimes \mathbb{1}, \quad \mathcal{U} c_x^* \otimes \mathbb{1} \mathcal{U}^* = \gamma(x) c_x \otimes \mathbb{1}, \quad \mathcal{U} \mathbb{1} \otimes c_x \mathcal{U}^* = \mathbb{1} \otimes c_x, \quad (3.7)$$

where $\gamma(x) = 1$ for $x \in \Lambda_e$, $\gamma(x) = -1$ for $x \in \Lambda_o$.

Lemma 3.1 *Let $\widehat{H}_{e-p} = \mathcal{U}H_{e-p}\mathcal{U}^*$ and $\widehat{H}_{e-e} = \mathcal{U}H_{e-e}\mathcal{U}^*$. We have the following:*

- (i) $\widehat{H}_{e-p} = -T_{-e,\uparrow} - T_{+e,\downarrow}$.
- (ii) $\widehat{H}_{e-e} = \frac{1}{2} \sum_{x,y \in \Lambda} U(x-y)(\mathbf{n}_x \otimes \mathbb{1} - \mathbb{1} \otimes \mathbf{n}_x)(\mathbf{n}_y \otimes \mathbb{1} - \mathbb{1} \otimes \mathbf{n}_y)$.

Proof. (i) By the definition of \mathcal{U} , we have

$$\mathcal{U}T_{+e,\uparrow}\mathcal{U}^* = \sum_{x,y \in \Lambda, |x-y|=1} t\gamma(x)\gamma(y)c_x c_y^* \otimes \mathbb{1} \otimes \exp \left\{ ie \int_{C_{xy}} dr \cdot A(r) \right\}. \quad (3.8)$$

Because $\gamma(x)\gamma(y) = -1$ holds provided $|x-y| = 1$, we have

$$\begin{aligned} &= \sum_{x,y \in \Lambda, |x-y|=1} t c_y^* c_x \otimes \mathbb{1} \otimes \exp \left\{ ie \int_{C_{xy}} dr \cdot A(r) \right\} \\ &= \sum_{x,y \in \Lambda, |x-y|=1} t c_x^* c_y \otimes \mathbb{1} \otimes \exp \left\{ -ie \int_{C_{xy}} dr \cdot A(r) \right\} \\ &= T_{-e,\uparrow}. \end{aligned} \quad (3.9)$$

Here we used that $\int_{C_{yx}} dr \cdot A(r) = -\int_{C_{xy}} dr \cdot A(r)$. Similarly one obtains $\mathcal{U}T_{+e}\mathcal{U}^* = T_{+e}$. (ii) is obvious. \square

Now we arrive at the following expression:

$$\mathcal{U}H\mathcal{U}^* = \widehat{H}, \quad (3.10)$$

$$\widehat{H} = \widehat{H}_{e-p} + \widehat{H}_{e-e} \otimes \mathbb{1} \otimes \mathbb{1} + \mathbb{1} \otimes \mathbb{1} \otimes H_f, \quad (3.11)$$

where \widehat{H}_{e-p} and \widehat{H}_{e-e} are defined in Lemma 3.1.

3.2 Gaussian domination

For each $\mathbf{h} = \{h_x\}_{x \in \Lambda} \in \mathbb{R}^{|\Lambda|}$, let $\widehat{H}(\mathbf{h})$ be the Hamiltonian \widehat{H} with

$$H_{e-e}(\mathbf{h}) = \frac{1}{2} \sum_{x,y \in \Lambda} U(x-y)(\mathbf{n}_x \otimes \mathbb{1} - \mathbb{1} \otimes \mathbf{n}_x - h_x)(\mathbf{n}_y \otimes \mathbb{1} - \mathbb{1} \otimes \mathbf{n}_y - h_y). \quad (3.12)$$

Note that $\widehat{H} = \widehat{H}(\mathbf{0})$ holds.

Under the identification $\mathfrak{E} \otimes \mathfrak{P} = (\mathfrak{F}_a \otimes \mathfrak{F}_a) \otimes L^2(\mathcal{Q}) = \int_{\mathcal{Q}}^{\oplus} \mathfrak{F}_a \otimes \mathfrak{F}_a d\nu$, we have

$$\widehat{H}_{e-p} = - \int_{\mathcal{Q}}^{\oplus} T_{-e,\uparrow}(\phi) d\nu(\phi) - \int_{\mathcal{Q}}^{\oplus} T_{+e,\downarrow}(\phi) d\nu(\phi), \quad (3.13)$$

where

$$T_{\pm e,\uparrow}(\phi) = \sum_{x,y \in \Lambda, |x-y|=1} t \exp \left\{ \pm ie \int_{C_{xy}} dr \cdot \mathcal{A}(r)(\phi) \right\} c_x^* c_y \otimes \mathbb{1}, \quad (3.14)$$

$$T_{\pm e,\downarrow}(\phi) = \sum_{x,y \in \Lambda, |x-y|=1} t \exp \left\{ \pm ie \int_{C_{xy}} dr \cdot \mathcal{A}(r)(\phi) \right\} \mathbb{1} \otimes c_x^* c_y, \quad (3.15)$$

and the vector potential $\mathcal{A}(x)$ is given by $\mathcal{A}(x) = \mathcal{A}(\oplus^3 \rho(\cdot - x))$ with $\rho = (\omega^{-1/2} \chi_\kappa)^\vee$, see (2.13). Here \check{f} is the inverse Fourier transformation of $f \in \ell^2(V^*)$.

Let $K = \widehat{H}_{e-p} + d\Gamma(\tilde{\omega})$ and let

$$\mathcal{Z}_{\beta,n,\varepsilon}(\mathbf{h}) = \text{Tr} \left[\left(e^{-\beta K/n} e^{-\beta \widehat{H}_{e-e}(\mathbf{h})/n} \right)^n e^{-\varepsilon d\Gamma(\tilde{\omega})} \right], \quad n \in \mathbb{N}, \quad \varepsilon > 0. \quad (3.16)$$

Lemma 3.2 *Let $T(\Phi_s) = T_{-e,\uparrow}(\Phi_s) + T_{+e,\downarrow}(\Phi_s)$, where $T_{\pm e,\sigma}(\Phi_s)$ is given by (3.14) and (3.15) with $\mathcal{A}(r)(\Phi_s)$, $\Phi \in \mathcal{Q}_\beta$. One has*

$$\begin{aligned} & \mathcal{Z}_{\beta,n,\varepsilon}(\mathbf{h}) / Z_{\beta+\varepsilon} \\ &= (4\pi)^{-n|\Lambda|/2} \int_{\mathbb{R}^{n|\Lambda|}} \prod_{j=1}^n d\mathbf{k}_j \int_{\mathcal{Q}_{\beta+\varepsilon}} d\nu_{\beta+\varepsilon}(\Phi) e^{-i \sum_{j=1}^n \mathbf{h} \cdot \mathbf{k}_j} e^{-\sum_{j=1}^n \mathbf{k}_j^2/4} \\ & \quad \times \text{Tr}_{\mathfrak{F}_a \otimes \mathfrak{F}_a} \left[\prod_{j=1}^n \left\{ \left(\prod_{(j-1)\beta/n}^{j\beta/n} e^{T(\Phi_s)ds} \right) \exp \left\{ i \sum_{j=1}^n \sum_{x,y \in \Lambda} \frac{\beta}{2n} k_{jx} U(x-y) (\mathbf{n}_y \otimes \mathbb{1} - \mathbb{1} \otimes \mathbf{n}_y) \right\} \right\} \right], \end{aligned} \quad (3.17)$$

where $\prod_{j=1}^m A_j := A_1 A_2 \cdots A_m$, the ordered product, and $\prod_{(j-1)\beta/n}^{j\beta/n} e^{T(\Phi_s)ds}$ is the strong product integration defined by (3.20) below.

Proof. By the Trotter-Kato product formula, Lemma 2.1 and Lemma 3.5 below, we have

$$\begin{aligned} & \mathcal{Z}_{\beta,n,\varepsilon}(\mathbf{h}) / Z_{\beta+\varepsilon} \quad (3.18) \\ &= \frac{1}{Z_{\beta+\varepsilon}} \lim_{M_1 \rightarrow \infty} \cdots \lim_{M_n \rightarrow \infty} \text{Tr}_{\mathfrak{H}_M} \left[\left(e^{-\beta d\Gamma(\tilde{\omega})/nM_1} e^{-\beta \widehat{H}_{e-p}/nM_1} \right)^{M_1} e^{-\beta \widehat{H}_{e-e}(\mathbf{h})/n} \times \right. \\ & \quad \left. \cdots \times \left(e^{-\beta d\Gamma(\tilde{\omega})/nM_n} e^{-\beta \widehat{H}_{e-p}/nM_n} \right)^{M_n} e^{-\beta \widehat{H}_{e-e}(\mathbf{h})/n} e^{-\varepsilon d\Gamma(\tilde{\omega})} \right] \\ &= \lim_{M_1 \rightarrow \infty} \cdots \lim_{M_n \rightarrow \infty} \int_{\mathcal{Q}_{\beta+\varepsilon}} d\nu_{\beta+\varepsilon}(\Phi) \text{Tr}_{\mathfrak{F}_a \otimes \mathfrak{F}_a} \left[\left(\prod_{j=1}^{M_1} \exp \left\{ \frac{\beta}{nM_1} T \left(\Phi_{\frac{j}{nM_1}\beta} \right) \right\} \right) e^{-\beta \widehat{H}_{e-e}(\mathbf{h})/n} \right. \\ & \quad \times \left(\prod_{j=1}^{M_2} \exp \left\{ \frac{\beta}{nM_2} T \left(\Phi_{\frac{1}{n}\beta + \frac{j}{nM_1}\beta} \right) \right\} \right) e^{-\beta \widehat{H}_{e-e}(\mathbf{h})/n} \\ & \quad \left. \cdots \times \left(\prod_{j=1}^{M_n} \exp \left\{ \frac{\beta}{nM_n} T \left(\Phi_{\frac{n-1}{n}\beta + \frac{j}{nM_n}\beta} \right) \right\} \right) e^{-\beta \widehat{H}_{e-e}(\mathbf{h})/n} \right]. \end{aligned} \quad (3.19)$$

Note that $T(\Phi_s)$ is continuous in s for each $\Phi \in \mathcal{Q}_\beta$. Thus the following strong product

integration exists [3]:

$$\text{s-}\lim_{M \rightarrow \infty} \prod_{j=1}^{\overrightarrow{M}} \exp \left\{ \frac{\beta}{nM} T \left(\Phi_{s + \frac{j}{nM} \beta} \right) \right\} =: \prod_s^{\overrightarrow{s + \frac{\beta}{n}}} e^{T(\Phi_s) ds}, \quad (3.20)$$

where s-lim means the strong limit. Thus we see that the R.H.S. of (3.19) converges to

$$\begin{aligned} \int_{\mathcal{Q}_{\beta+\varepsilon}} d\nu_{\beta+\varepsilon}(\Phi) \text{Tr}_{\mathfrak{F}_a \otimes \mathfrak{F}_a} & \left[\left(\prod_0^{\overrightarrow{\frac{\beta}{n}}} e^{T(\Phi_s) ds} \right) e^{-\beta \hat{H}_{e-e}(\mathbf{h})/n} \times \right. \\ & \left. \dots \times \left(\prod_{\frac{n-1}{n}\beta}^{\overrightarrow{\beta}} e^{T(\Phi_s) ds} \right) e^{-\beta \hat{H}_{e-e}(\mathbf{h})/n} \right]. \end{aligned} \quad (3.21)$$

Note that since $U(x-y)$ is a positive semidefinite matrix, it holds that

$$\begin{aligned} & e^{-\beta \hat{H}_{e-e}(\mathbf{h})/n} \\ & = (4\pi)^{-|\Lambda|/2} \int_{\mathbb{R}^{|\Lambda|}} d\mathbf{k} e^{-i\mathbf{h} \cdot \mathbf{k}} e^{-\mathbf{k}^2/4} \exp \left\{ i \sum_{x,y \in \Lambda} \frac{\beta}{2n} U(x-y) k_x (\mathbf{n}_y \otimes \mathbb{1} - \mathbb{1} \otimes \mathbf{n}_y) \right\}. \end{aligned} \quad (3.22)$$

Now we obtain the assertion in the lemma. \square

Proposition 3.3 *For each $\Phi \in \mathcal{Q}_\beta$, let $\mathbb{T}_{\pm e}(\Phi_s)$ be given by*

$$\mathbb{T}_{\pm e}(\Phi_s) = \sum_{x,y \in \Lambda, |x-y|=1} t \exp \left\{ \pm i e \int_{C_{xy}} dr \cdot \mathcal{A}(r)(\Phi_s) \right\} c_x^* c_y. \quad (3.23)$$

One has the following:

$$\begin{aligned} & \mathcal{Z}_{\beta,n,\varepsilon}(\mathbf{h}) / Z_{\beta+\varepsilon} \\ & = (4\pi)^{-n|\Lambda|/2} \int_{\mathbb{R}^{n|\Lambda|}} \prod_{j=1}^n d\mathbf{k}_j \int_{\mathcal{Q}_{\beta+\varepsilon}} d\nu_{\beta+\varepsilon}(\Phi) e^{-i \sum_{j=1}^n \mathbf{k}_j \cdot \mathbf{h}} e^{-\sum_{j=1}^n \mathbf{k}_j^2/4} \\ & \quad \times \left| \text{Tr}_{\mathfrak{F}_a} \left[\prod_{j=1}^n \left(\prod_{(j-1)\beta/n}^{\overrightarrow{j\beta/n}} e^{\mathbb{T}_{-e}(\Phi_s) ds} e^{i \sum_{j=1}^n \sum_{x,y \in \Lambda} \frac{\beta}{2n} k_{jx} U(x-y) \mathbf{n}_y} \right) \right] \right|^2. \end{aligned} \quad (3.24)$$

Proof. By the fact $\text{Tr}[A \otimes B] = \text{Tr}[A]\text{Tr}[B]$ and Lemma 3.2, we obtain

$$\begin{aligned}
& \mathcal{Z}_{\beta,n,\varepsilon}(\mathbf{h}) / Z_{\beta+\varepsilon} \\
&= (4\pi)^{-n|\Lambda|/2} \int_{\mathbb{R}^{n|\Lambda|}} \prod_{j=1}^n d\mathbf{k}_j \int_{\mathcal{Q}_{\beta+\varepsilon}} d\nu_{\beta+\varepsilon}(\Phi) e^{-i \sum_{j=1}^n \mathbf{k}_j \cdot \mathbf{h}} e^{-\sum_{j=1}^n \mathbf{k}_j^2/4} \\
&\times \text{Tr}_{\mathfrak{F}_a} \left[\prod_{j=1}^n \left(\prod_{(j-1)\beta/n}^{j\beta/n} e^{\mathbb{T}_{-e}(\Phi_s)ds} e^{+i \sum_{j=1}^n \sum_{x,y \in \Lambda} \frac{\beta}{2n} k_{jx} U(x-y) \mathbf{n}_y} \right) \right] \\
&\times \text{Tr}_{\mathfrak{F}_a} \left[\prod_{j=1}^n \left(\prod_{(j-1)\beta/n}^{j\beta/n} e^{\mathbb{T}_{+e}(\Phi_s)ds} e^{-i \sum_{j=1}^n \sum_{x,y \in \Lambda} \frac{\beta}{2n} k_{jx} U(x-y) \mathbf{n}_y} \right) \right]. \tag{3.25}
\end{aligned}$$

Let Θ be a conjugation in \mathfrak{F}_a defined by $\Theta c_{x_1}^* \cdots c_{x_N}^* \Omega_a = c_{x_1}^* \cdots c_{x_N}^* \Omega_a$, where Ω_a is the Fock vacuum in \mathfrak{F}_a . Noting that $\Theta c_x \Theta = c_x$, we have $\Theta \mathbb{T}_{+e}(\Phi_s) \Theta = \mathbb{T}_{-e}(\Phi_s)$ and $\Theta \mathbf{n}_x \Theta = \mathbf{n}_x$. Thus it holds that

$$\Theta \prod_{(j-1)\beta/n}^{j\beta/n} e^{\mathbb{T}_{+e}(\Phi_s)ds} \Theta = \prod_{(j-1)\beta/n}^{j\beta/n} e^{\mathbb{T}_{-e}(\Phi_s)ds}, \tag{3.26}$$

$$\Theta e^{-i \sum_{j=1}^n \sum_{x,y \in \Lambda} \frac{\beta}{2n} k_{jx} U(x-y) \mathbf{n}_y} \Theta = e^{+i \sum_{j=1}^n \sum_{x,y \in \Lambda} \frac{\beta}{2n} k_{jx} U(x-y) \mathbf{n}_y}. \tag{3.27}$$

Hence using the fact $\text{Tr}[A] = (\text{Tr}[\Theta A \Theta])^*$, one observes that

$$\begin{aligned}
& \text{Tr}_{\mathfrak{F}_a} \left[\prod_{j=1}^n \left(\prod_{(j-1)\beta/n}^{j\beta/n} e^{\mathbb{T}_{+e}(\Phi_s)} e^{-i \sum_{j=1}^n \sum_{x,y \in \Lambda} \frac{\beta}{2n} k_{jx} U(x-y) \mathbf{n}_y} \right) \right] \\
&= \left\{ \text{Tr}_{\mathfrak{F}_a} \left[\Theta \prod_{j=1}^n \left(\prod_{(j-1)\beta/n}^{j\beta/n} e^{\mathbb{T}_{+e}(\Phi_s)} e^{-i \sum_{j=1}^n \sum_{x,y \in \Lambda} \frac{\beta}{2n} k_{jx} U(x-y) \mathbf{n}_y} \right) \Theta \right] \right\}^* \\
&= \left\{ \text{Tr}_{\mathfrak{F}_a} \left[\prod_{j=1}^n \left(\prod_{(j-1)\beta/n}^{j\beta/n} e^{\mathbb{T}_{-e}(\Phi_s)} e^{+i \sum_{j=1}^n \sum_{x,y \in \Lambda} \frac{\beta}{2n} k_{jx} U(x-y) \mathbf{n}_y} \right) \right] \right\}^*. \tag{3.28}
\end{aligned}$$

This completes the proof. \square

Theorem 3.4 *Let $\mathcal{Z}_\beta(\mathbf{h}) = \text{Tr}[e^{-\beta \hat{H}(\mathbf{h})}]$. Then for all $\mathbf{h} \in \mathbb{R}^{|\Lambda|}$, $\mathcal{Z}_\beta(\mathbf{h}) \leq \mathcal{Z}_\beta(\mathbf{0})$ holds.*

Proof. By Proposition 3.3, it holds that $|\mathcal{Z}_{\beta,n,\varepsilon}(\mathbf{h})| \leq \mathcal{Z}_{\beta,n,\varepsilon}(\mathbf{0})$. As $n \rightarrow \infty$, $\mathcal{Z}_{\beta,n,\varepsilon}(\mathbf{h})$ converges to $\mathcal{Z}_{\beta,\varepsilon}(\mathbf{h}) = \text{Tr}[e^{-\beta \hat{H}(\mathbf{h})} e^{-\varepsilon d\Gamma(\tilde{\omega})}]$ by Lemma 3.5. Thus we have $|\mathcal{Z}_{\beta,\varepsilon}(\mathbf{h})| \leq \mathcal{Z}_{\beta,\varepsilon}(\mathbf{0})$. As $\varepsilon \downarrow 0$, $\mathcal{Z}_{\beta,\varepsilon}(\mathbf{h})$ converges to $\mathcal{Z}_\beta(\mathbf{h})$ by Lemma 3.5. \square

Lemma 3.5 *We denote by $L^1(\mathfrak{X})$ the ideal of all trace class operators on \mathfrak{X} . Let $A_n, A \in L^\infty(\mathfrak{X})$ and $B_n, B \in L^1(\mathfrak{X})$ such that A_n converges to A strongly and $\|B_n - B\|_1 \rightarrow 0$ as $n \rightarrow \infty$, where $\|\cdot\|_1$ is the trace norm. Then $\|A_n B_n - AB\|_1 \rightarrow 0$ as $n \rightarrow \infty$.*

Proof. See [19, Chap. 2, Example 3]. \square

Corollary 3.6 *For all $\mathbf{h} \in \mathbb{C}^{|\Lambda|}$, we have*

$$\left(\langle \delta \mathbf{n}, \mathbf{U} \mathbf{h} \rangle^*, \langle \delta \mathbf{n}, \mathbf{U} \mathbf{h} \rangle \right)_{\beta, \Lambda} \leq \beta^{-1} \langle \mathbf{h}, \mathbf{U} \mathbf{h} \rangle, \quad (3.29)$$

where $\langle \delta \mathbf{n}, \mathbf{U} \mathbf{h} \rangle := \sum_{x, y \in \Lambda} U(x - y) \delta n_x h_y$ and $\langle \mathbf{h}, \mathbf{U} \mathbf{h} \rangle := \sum_{x, y \in \Lambda} h_x^* U(x - y) h_y$.

Proof. Let $((A, B))_{\beta, \Lambda}$ be the Duhamel two-point function associated with \hat{H} . Then by Theorem 3.4, we have

$$\left(\left(\langle \mathbf{q}, \mathbf{U} \mathbf{h} \rangle^*, \langle \mathbf{q}, \mathbf{U} \mathbf{h} \rangle \right) \right)_{\beta, \Lambda} \leq \beta^{-1} \langle \mathbf{h}, \mathbf{U} \mathbf{h} \rangle, \quad (3.30)$$

where $q_x := n_x \otimes \mathbb{1} - \mathbb{1} \otimes n_x$. Since $((\langle \mathbf{q}, \mathbf{U} \mathbf{h} \rangle^*, \langle \mathbf{q}, \mathbf{U} \mathbf{h} \rangle))_{\beta, \Lambda} = ((\langle \delta \mathbf{n}, \mathbf{U} \mathbf{h} \rangle^*, \langle \delta \mathbf{n}, \mathbf{U} \mathbf{h} \rangle))_{\beta, \Lambda}$, we get the result in the corollary. \square

3.3 Completion of proof of Theorem 1.2

By Corollary 3.6, we obtain

$$\beta (\widetilde{\delta n_{-p}}, \widetilde{\delta n_p})_{\beta, \Lambda} \leq \hat{U}_\Lambda(p)^{-1}. \quad (3.31)$$

Since $|\hat{U}_\Lambda(p) - \hat{U}(p)| \rightarrow 0$ as $\ell \rightarrow \infty$ by **(A. 1)**, we obtain the desired result. \square

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